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TOWARDS A UNIFIED RATE THEORY OF STOCHASTIC RESONANCE

MATTHEW R. BENNETT
Institute for Nonlinear Science and Department of Bioengineering,
University of California, San Diego
Mailcode 0412, 9500 Gilman Dr., La Jolla, CA 92093-0412

KURT WIESENFELD
Center for Nonlinear Science and School of Physics, Georgia Institute of Technology
837 State Street, Atlanta, GA 30332-0430

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We present a theory which connects two-state and excitable versions of stochastic resonance. The latter appears as an extreme asymmetric limit of traditional two-state rate theory. To achieve this unified view we are led to a simple generalization of excitable stochastic resonance.

Keywords: Stochastic resonance; excitable dynamics.

1. Introduction

Stochastic resonance (SR) has a rich history that dates back 25 years [1,2]. It was originally proposed as a mechanism for the observed periodicity of Earth's ice ages, and although the data has not supported the idea that Earth's major Ice Ages are a consequence of SR [3], research into the climatic relevance of SR continues [4]. Furthermore, SR has long since been shown to have a wide range of applicability [5].

Several distinct mechanisms for stochastic resonance exist [6,7], but two of these (two-state and excitable) are the most widely studied. In the case of two-state SR, the system can reside in one of two deterministically stable states. Noise induced activated escape causes the system to switch between these two states, while a periodic signal biases the system first towards one state and then the other. In the weak signal limit, the output signal to noise ratio (SNR) can show a maximum at a non-zero input noise level. In contrast, excitable systems can stably reside in just one state. Activated escape causes the system to leave this state and enter a transient "excited" phase that quickly (and perhaps deterministically) subsides as the system returns to the stable state. The excited phase is assumed to generate a standard pulse at the output. A periodic signal causes the random spike train to bunch and anti-bunch. Once again a maximum in the SNR may occur for a finite level of noise.

Though there are other types of theories that model SR [8,9], rate theories are an intuitive way of understanding the processes behind the phenomenon. Existing rate theories for the two versions of SR in question are quite different: whereas two state theory involves hopping between states as typically described by rate equations, excitable analysis treats the statistics of a (time-dependent) Poissonian spike train. Despite the distinct underlying pictures and analyses, the specific predictions for the output signal to noise ratios are intriguingly similar [10, 11]. This coincidence has led to speculation about whether a single theory could encompass both types of SR [6]. In this paper, we present such a unified description.

Our unified treatment closely follows the standard two state rate theory of stochastic resonance [10, 12]. In the extreme asymmetric limit we recover a version of the excitable theory. Unlike the canonical excitable model [11], the excitable limit of the two state model will not have uniform pulses. Instead, the widths of the pulses are exponentially distributed. We will account for this non-uniformity by first making a straightforward generalization of the existing theory for excitable systems. Then, we will make the connection between the two types of theories with a model that is correct in both the symmetric two state and excitable limits.

2. Generalized Excitable SR

Consider a sequence of pulses generated by the time-dependent rate $\alpha(t)$. If each pulse has the same shape $F(t)$, the autocorrelation function for the pulse train is [13, 14]

$$\psi(\tau,t) = \alpha(t)\int_0^\infty F(t)F(t+\tau)\mathrm{d}t + \alpha(t)\alpha(t+\tau)\left\{\int_0^\infty F(t)\mathrm{d}t\right\}^2, \tag{1}$$

If each pulse is rectangular, with height A and width Δt, then the first integral in Eq. (1) becomes

$$C_0(\tau;\Delta t) = \int_0^\infty F(t)F(t+\tau)\mathrm{d}t = A^2\Delta t\left(1-\frac{|\tau|}{\Delta t}\right) \tag{2}$$

for $|\tau| < \Delta t$ and $C_0(\tau) = 0$ otherwise.

Our generalization is to allow the widths of each pulse to be distributed randomly according to some probability distribution, $P(\Delta t)$. Instead of Eq. (1) we have

$$\begin{aligned}\psi(\tau,t) &= \alpha(t)\int_0^\infty \mathrm{d}(\Delta t)P(\Delta t)C_0(\tau;\Delta t)\\ &\quad + \alpha(t)\alpha(t+\tau)\left\{\int_0^\infty \mathrm{d}(\Delta t)P(\Delta t)\int_0^\infty F(t)\mathrm{d}t\right\}^2.\end{aligned} \tag{3}$$

The integrals in the second term on the R.H.S. of Eq. (3) represent the area of the pulse averaged over the distribution of pulse widths. Thus,

$$\psi(\tau,t) = \alpha(t)K(\tau) + \alpha(t)\alpha(t+\tau)A^2\langle\Delta t\rangle^2, \tag{4}$$

where $\langle \Delta t \rangle$ is the average pulse width and

$$K(\tau) = \int_0^\infty \mathrm{d}(\Delta t) P(\Delta t) C_0(\tau; \Delta t). \tag{5}$$

Following Ref. [11] we take the time dependent rate to be $\alpha(t) = \alpha + \epsilon \cos(\omega_s t + \phi)$, so that

$$\psi(\tau) = \{\psi(\tau, t)\}_\phi = \alpha K(\tau) + \left[\alpha^2 + \frac{1}{2}\epsilon^2 \cos(\omega_s \tau)\right] A^2 \langle \Delta t \rangle^2, \tag{6}$$

where $\{\ldots\}_\phi$ represents an average over the initial phase of the drive. To obtain the power spectrum, we take the one-sided cosine transform [13] of $\psi(\tau)$ to arrive at

$$S(\omega > 0) = \alpha \tilde{K}(\omega) + \pi \epsilon^2 A^2 \langle \Delta t \rangle^2 \delta(\omega - \omega_s), \tag{7}$$

where $\tilde{K}(\omega)$ is the Fourier transform of $K(\tau)$. The first term represents broadband noise while the second is the power of the coherent signal at $\omega = \omega_s$. The SNR is defined to be the ratio of the coefficient of the delta function to the power of the broadband noise at the signal frequency, which is

$$R = \frac{\pi \epsilon^2 A^2 \langle \Delta t \rangle^2}{\alpha \tilde{K}(\omega_s)}. \tag{8}$$

Equation (8) allows us to calculate the SNR for any pulse train that has a firing rate $\alpha(t)$ and whose pulse widths are distributed according to a known probability distribution. We are particulary interested in two distinct cases. The first is the "fixed-width" limit in which each pulse is identical. This corresponds to the situation used in the theory of excitable SR developed in Ref. [11]. The second case is where we will allow the pulse widths to vary according to an exponential distribution. It is this second case limit that will allow us, in the next section, to make the connection between two-state and excitable SR.

In the fixed pulse-width case the probability distribution of the widths is $P(\Delta t) = \delta(\Delta t - \langle \Delta t \rangle)$. For the excitable dynamics we are trying to model, the width of each pulse is very small. We therefore take the double limit $\langle \Delta t \rangle \to 0$ and $A \to \infty$ such that the product $A\langle \Delta t \rangle$ remains constant (representing the "area" of each pulse), which guarantees an output that has nonzero power. In the fixed-width limit $K(\tau)$ reduces to

$$K(\tau) = A^2 \langle \Delta t \rangle^2 \delta(\tau) \tag{9}$$

and its transform is

$$\tilde{K}(\omega) = 2A^2 \langle \Delta t \rangle^2. \tag{10}$$

The resulting SNR is

$$R = \frac{\pi \epsilon^2}{2\alpha}, \tag{11}$$

which is the SNR calculated in Ref. [11] for SR in excitable systems[1].

[1]Our result differs from the result of Ref. [11] by a factor of $\pi/2$. This is due to their choice for the definition of the SNR. Also note that Eq. (7) of their paper has an inadvertent factor of $1/2$ that is corrected later in the paper.

Next, consider the case of exponentially distributed pulse widths. In particular, let the probability distribution be of the form

$$P(\Delta t) = \frac{1}{\langle \Delta t \rangle} \exp\left[-\frac{\Delta t}{\langle \Delta t \rangle}\right]. \tag{12}$$

From Eqs. (2) and (5),

$$\begin{aligned} K(\tau) &= \frac{A^2}{\langle \Delta t \rangle} \int_0^\infty \mathrm{d}(\Delta t) \exp\left[-\frac{\Delta t}{\langle \Delta t \rangle}\right] \Delta t \left(1 - \frac{|\tau|}{\Delta t}\right) Q \\ &= A^2 \langle \Delta t \rangle \exp\left[-\frac{|\tau|}{\langle \Delta t \rangle}\right], \end{aligned} \tag{13}$$

where $Q = 1$ if $|\tau| < \Delta t$ and is zero otherwise. If we again take the limit $\langle \Delta t \rangle \to 0$ while keeping $A\langle \Delta t \rangle$ constant we get

$$K(\tau) = 2A^2 \langle \Delta t \rangle^2 \delta(\tau), \tag{14}$$

and its transform

$$\tilde{K}(\omega) = 4A^2 \langle \Delta t \rangle^2. \tag{15}$$

Notice that $\tilde{K}(\omega)$ is larger than its fixed width limit counterpart Eq. (10) by a factor of two. This represents an increase in the broadband noise resulting from the nonidentical pulse widths. The SNR is this case becomes

$$R = \frac{\pi \epsilon^2}{4\alpha}. \tag{16}$$

As mentioned earlier, the case of exponentially distributed pulse widths is of particular interest to us since it allows us to make a connection with two-state systems.

3. The Constrained Asymmetric Rate Model

In this section we show that stochastic resonance in excitable systems can be viewed as a limiting case of the asymmetric two state rate theory. Figure 1 illustrates the basic situation. A two state system generates an output time series that is similar to a telegraph signal [15], with random switching points determined by the two transition rates. We can associate the lower level with an "off" state, so that the time series represents a sequence of rectangular "on" pulses of amplitude A. Imagine that the rate out of the lower state is held fixed, while the rate out of the upper state is made larger and larger. Then the widths Δt_i of the rectangular pulses diminish (on average); in the extreme case the time series resembles the spike-train output of an excitable system. In taking this limit, we simultaneously increase the state value A in order to get a finite output power.

Following Ref. [12], we start with the rate equation

$$\dot{P}_+ = -W_+ P_+ + W_- P_- \tag{17}$$

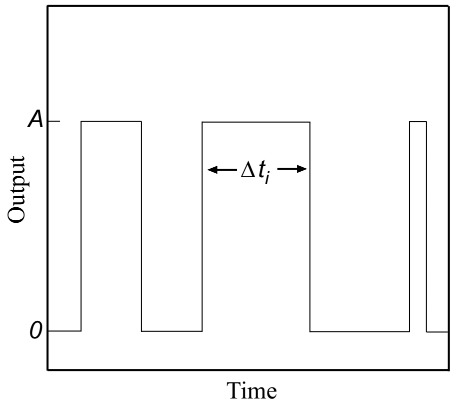

Fig. 1. Time series of the two state system. The random pulse widths Δt_i have mean value determined by the transition rate W_-. In the excitable limit both W_- and A tend to infinity.

and $P_+ + P_- = 1$ where $P_\pm$ is the probability that the system is in the upper/lower state, respectively, and the escape rates $W_\pm$ are

$$W_\pm(t) = \alpha_\pm \pm \epsilon_\pm \cos(\omega_s t + \phi)\,. \tag{18}$$

If one were to derive these rates, say from some underlying bistable Langevin model, the various rate parameters $\alpha_\pm, \epsilon_\pm$ would depend on the asymmetry in the barrier heights in the two wells. As the asymmetry increases, the difference $\Delta\epsilon = |\epsilon_+ - \epsilon_-|$ also increases. This is inconvenient for ensuing analysis because the resulting rate equations can be solved only up to quadrature [12, 16].

For our purposes, we are especially interested in exploring the limit where the asymmetry can be large (even infinite). We can do this in a way which greatly simplifies the intermediate calculations, by constraining the rate parameters in a particular way, i.e. $\epsilon_+ = \epsilon_- = \epsilon$. For this "constrained asymmetric rate model," the probability that the system is in the positive (+) state is governed by the differential equation

$$\dot{P}_+ = -\sigma P_+ + \alpha_- - \epsilon \cos(\omega_s t + \phi) \tag{19}$$

where $\sigma = \alpha_+ + \alpha_-$. Equation (19) has as its solution

$$\begin{aligned} P_+(t|s_0, t_0) \quad &= \quad \exp[-\sigma|t - t_0|] \left\{ \delta_{s_0,s_+} - \frac{\alpha_-}{\sigma} + \frac{\epsilon}{\sqrt{\sigma^2 + {\omega_s}^2}} \sin(\omega_s t + \theta) \right\} \ldots \\ &\quad \ldots + \frac{\alpha_-}{\sigma} - \frac{\epsilon}{\sqrt{\sigma^2 + {\omega_s}^2}} \sin(\omega_s t + \theta)\,, \end{aligned} \tag{20}$$

where $P_+(t|s_0, t_0)$ is the conditional probability that the system is in the positive state (s_+) at time t given that it was in state $s_0 \in \{s_+, s_-\}$ at time t_0, $\theta = \phi + \tan^{-1}(\sigma/\omega_s)$ and $\delta_{i,j}$ is the Kronecker delta.

We define the output I of the system to be A for state s_+ and zero otherwise. Equation (20) allows us to construct the autocorrelation function, defined as

$$\psi(\tau) = \lim_{t_0 \to \infty} \langle I(t+\tau) I(t) | s_0, t_0 \rangle\,, \tag{21}$$

where the brackets indicate averages over both the ensemble and the initial phase of the drive. The autocorrelation function and the resulting one-sided power spectrum are

$$\begin{aligned}\psi(\tau) \quad &= \quad \frac{A^2\alpha_-}{\sigma}\left[1-\frac{\alpha_-}{\sigma}\right]\exp\left(-\sigma|\tau|\right)-\frac{A^2\epsilon^2}{2(\sigma^2+{\omega_s}^2)}\exp\left(-\sigma|\tau|\right)\ldots \\ &\qquad \ldots+\frac{A^2{\alpha_-}^2}{\sigma^2}+\frac{A^2\epsilon^2}{2(\sigma^2+{\omega_s}^2)}\cos\left(\omega_s\tau\right)\end{aligned} \tag{22}$$

and

$$S(\omega>0)=\frac{4A^2\alpha_-}{\sigma^2+\omega^2}\left[1-\frac{\alpha_-}{\sigma}\right]-\frac{2A^2\epsilon^2\sigma}{(\sigma^2+{\omega_s}^2)(\sigma^2+\omega^2)}+\frac{\pi A^2\epsilon^2}{\sigma^2+{\omega_s}^2}\delta(\omega-\omega_s). \tag{23}$$

We can now read off the signal to noise ratio, with result

$$R=\pi\epsilon^2\left[4\alpha_-\left(1-\frac{\alpha_-}{\sigma}\right)-\frac{2\epsilon^2\sigma}{\sigma^2+{\omega_s}^2}\right]^{-1}. \tag{24}$$

Let's consider two limits. The first is the symmetric limit in which $\alpha_+ = \alpha_- = \alpha$. This limit is the classic two-state version of SR, and Eq. (24) reduces to

$$R=\frac{\pi\epsilon^2}{2\alpha}\left\{1-\frac{2\epsilon^2}{4\alpha^2+{\omega_s}^2}\right\}^{-1}, \tag{25}$$

which agrees with the SNR calculated in Ref. [10].

Now consider the excitable limit, in which α_+ is extremely large, so that the system will reside in the upper state for only a brief time. In the extreme limit we take $\alpha_+ \to \infty$ with α_- fixed, so that $\sigma = \alpha_+ + \alpha_- \to \alpha_+$, and Eq. (24) becomes

$$R=\frac{\pi\epsilon^2}{4\alpha_-}, \tag{26}$$

which is precisely the result we found earlier for an excitable system with exponentially distributed pulse widths, Eq. (16).

These two cases demonstrate that the constrained asymmetric rate model represents a unified description which recovers the SNR for both the classic two-state model and an excitable system in which the pulse widths are exponentially distributed.

4. Discussion

We have shown how a single rate theory is capable of describing both two-state and excitable stochastic resonance. In one limit, we recover the original theory for two state systems; in another limit, the "one-state", excitable version. Viewed in this light, the long-known similarities of the predicted SNRs for these two classes is understood in a natural way, rather than as an apparent coincidence.

To achieve this unified description, we had to generalize the existing framework for excitable systems to allow for non-uniform pulse trains. The asymmetric two-state model represents one type (with exponentially distributed pulse widths), while

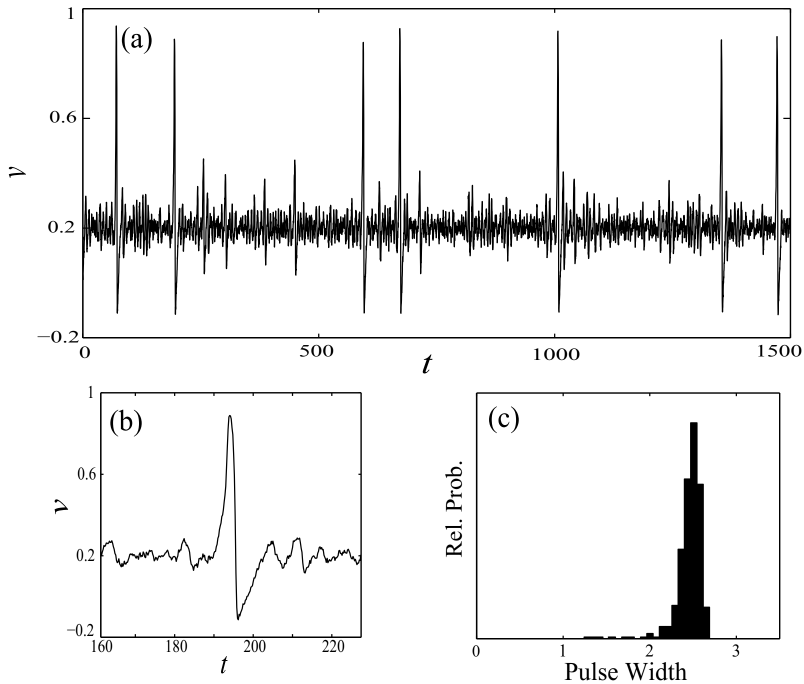

Fig. 2. (a) Representative time series of the Fitzhugh-Nagumo system. (b) Close up of an action potential. (c) Distribution of pulse widths has mean 2.46 and standard deviation 0.16. Parameter values: $I = 0.25, D = 0.06, \tau_c = 0.10, \tau_w = 10.0$.

the standard excitable model of SR represents another (with delta distributed pulse widths).

How relevant is this generalization to descriptions of actual excitable systems? It appears that it is necessary for some systems, while in other cases the old (uniform pulse) picture is adequate. As a simple illustration, we summarize in Figs. 2 and 3 the results of numerical simulations run on two different "canonical" excitable systems.

The first is the Fitzhugh-Nagumo equations

$$\tau_c \dot{v} = v(v - 0.5)(1 - v) - w + I + D\xi \tag{27}$$

$$\tau_w \dot{w} = v - w, \tag{28}$$

which represent a simple model sometimes used to describe spike-generation in neurons. It also happens to be one of the first reported examples of stochastic resonance in excitable systems [11,17]. Here, τ_c, τ_w, I and D are constants, and ξ is Gaussian (sampled) white noise with zero mean and unit variance. When I is small enough, the deterministic system has a stable fixed point, but noise can induce the coordinate v to display "action potential" spikes (Fig. 2a). The individual spikes

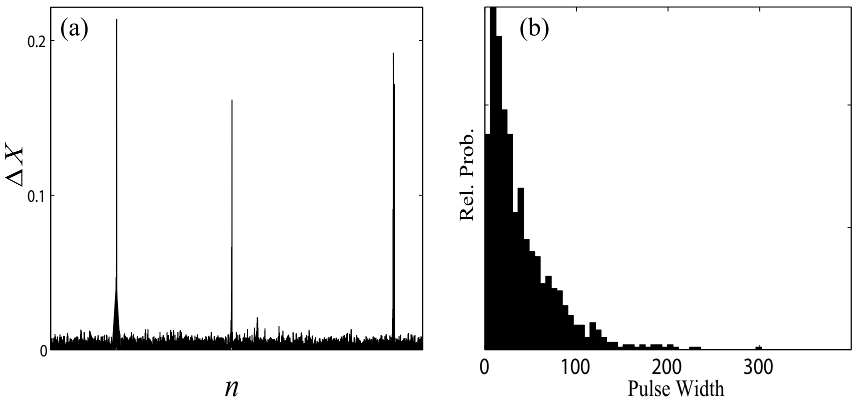

Fig. 3. (a) Time series of the Logistic map with $\lambda = 3.83$; plotted is the distance, ΔX_n, of X_n from the noise-free period-3 orbit. (b) In this simulation, bursts ranged from 5 to 300 iterates, and the distribution of pulse widths had a mean of 40 and standard deviation 37. The noise parameter was $D = 0.0015$. The duration of each burst was calculated by taking the difference between the indices at which the ΔX first becomes greater than 0.05 and when it first returns below 0.002. This histogram represents $10,000$ bursts.

have a characteristic line shape (Fig. 2b), even in the presence of noise, so that data extracted from a long time series yields a narrow distribution of pulse widths (Fig. 2c). Thus, this excitable system is reasonably described by a fixed pulse width theory. Real neurons also have very uniformly shaped spikes. While the amplitude and duration of spikes can show considerable variation between cells, the spikes from a single cell are very regular [18, 19].

Our second example is the excitable dynamics shown by the Logistic map operating within the stable period-3 window

$$X_{n+1} = \lambda X_n(1 - X_n) + D\xi_n; \qquad 0 \leq X_n \leq 1 \tag{29}$$

with $\lambda = 3.83$ and ξ_n a random variable uniformly distributed on $[-0.5, 0.5]$. The constant D sets the noise strength. When the perturbation due to noise is large enough, this system displays transient (chaotic) bursts (Fig. 3a) before settling back into the period-3 orbit. The duration of these bursts is rather broadly distributed (Fig. 3b), so that the fixed pulse-width theory underestimates the output noise power.

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References

[1] R. Benzi, A. Sutera and A. Vulpiani, Constructive role of noise in human brain activity, *Jour. of Phys. A* **14** (1981) L453–L457.

[2] C. Nicolis, Solar variability and the stochastic effects on climate, *Solar Physics* **74** (1981) 473–478.

[3] S. Rahmstorf and R. B. Alley, Stochastic resonance in glacial climate, *Eos* **83** (2002) 129–135.

[4] A. Ganopolski and S. Rahmstorf, Solar variability and the stochastic effects on climate, *Phys. Rev. Lett.* **88** (2002) 038501.

[5] L. Gammaitoni, P. Hänggi, P. Jung and F. Marchesoni, Stochastic resonance, *Rev. Mod. Phys.* **70** (1998) 223–287.

[6] K. Wiesenfeld and F. Jaramillo, Minireview of stochastic resonance, *Chaos* **8** (1998) 539–548.

[7] D. Pierson, D. O'Gorman and F. Moss, Stochastic resonance: Tutorial and update, *Intern. J. Bifurc. & Chaos* **4** (1994) 1383–1397.

[8] R. F. Fox, Stochastic Resonance in a double well, *Phys. Rev. A* **39** (1989) 4148–4153.

[9] B. Lindner and L. Schimansky-Geier, Coherence and stochastic resonance in a two-state system, *Phys. Rev. E* **61** (2000) 6103–6110.

[10] B. McNamara and K. Wiesenfeld, Theory of stochastic resonance, *Phys. Rev. A* **39** (1989) 4854–4869.

[11] K. Wiesenfeld, D. Pierson, E. Pantazelou, C. Dames and F. Moss, Stochastic resonance on a circle, *Phys. Rev. Lett.* **72** (1994) 2125–2129.

[12] R. Löfstedt and S. N. Coppersmith, Stochastic resonance: Nonperturbative calculation of power spectra and residence-time distributions, *Phys. Rev. E* **49** (1994) 4821–4831.

[13] S. O. Rice, Mathematical analysis of random noise, in *Selected Papers on Noise and Stochastic Processes*, ed. N. Wax (Dover, New York), (1954) 133–294.

[14] R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1963).

[15] D. R. Cox and P. A. W. Lewis, *The Statistical Analysis of Series of Events* (Wiley, New York 1966).

[16] J. Li, Effect of asymmetry on stochastic resonance and stochastic resonance induced by multiplicative noise and by mean-field coupling, *Phys. Rev. E* **66** (2002) 031104.

[17] R. L. Douglass, D. Pierson, L. A. Wilkens, E. Pantazelou and F. Moss, Stochastic resonance in an electronic Fitzhugh-Nagumo model, *Ann. New York Acad. Sci.* **706** (1993) 26.

[18] W. Gerstner, Time structure of the activity in neural network models, *Phys. Rev. E* **51** (1995) 738–758.

[19] W. Kistler, W. Gerstner and J.L. van Hemmen, Reduction of the Hodgkin-Huxley equations to a single-variable threshold model, *Neur. Comp.* **9** (1997) 1015–1045.